Linear Algebra II

06/05/2014, Monday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 $(8 + 10 = 18 \text{ pts})$ **Inner product spaces**

(a) Let V be an inner product space. Find real numbers a and b such that the so-called Apollonius' identity

$$\|z-x\|^2 + \|z-y\|^2 = a\|x-y\|^2 + b\|z - \frac{x+y}{2}\|^2$$

holds for any triple x, y, and z in V.

(b) Consider the vector space $C[-1,1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find the best approximation of the constant function 1 within the subspace spanned by the vectors x and $|x|$.

Required Knowledge: **inner product, Gram-Schmidt process**

Solution:

(1a):

The left hand side can be written as

$$2\langle z,z\rangle + \langle x,x\rangle + \langle y,y\rangle - \langle x,z\rangle - \langle z,x\rangle - \langle y,z\rangle - \langle z,y\rangle.$$

In addition the right hand side is

$$a(\langle x,x\rangle + \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle) + b(\langle z,z\rangle + \langle \frac{x+y}{2}, \frac{x+y}{2}\rangle - \langle z, \frac{x+y}{2}\rangle - \langle \frac{x+y}{2}, z\rangle)$$
$$= b\langle z,z\rangle + \left(\frac{1}{4}b + a\right)(\langle x,x\rangle + \langle y,y\rangle) + \left(\frac{1}{4}b - a\right)(\langle x,y\rangle + \langle y,x\rangle) - \frac{1}{2}b\left(\langle z,x\rangle + \langle x,z\rangle + \langle z,y\rangle + \langle y,z\rangle\right).$$

We get that the following should hold:

$$\begin{aligned} b &= 2 \\ \tfrac{1}{4}b + a &= 2 \\ \tfrac{1}{4}b - a &= 0 \\ \tfrac{1}{2}b &= 2, \end{aligned}$$

which only hold for $a = \frac{1}{2}$ and $b = 2$.

Remark. In real inner product space the left hand side is

$$2\langle z,z\rangle + \langle x,x\rangle + \langle y,y\rangle - 2\langle x,z\rangle - 2\langle y,z\rangle$$

and the right hand side is

$$= b\langle z,z\rangle + \left(\frac{1}{4}b + a\right)(\langle x,x\rangle + \langle y,y\rangle) + \left(\frac{1}{2}b - 2a\right)(\langle x,y\rangle) - b\left(\langle z,x\rangle + \langle z,y\rangle\right).$$

(1b): Note that

$$\langle x, |x| \rangle = 0.$$

As such, these two vectors are orthogonal. In order to obtain an orthonormal basis, we first compute the norms:

$$\|x\|^2 = \||x|\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{x^3}{3}|_{-1}^{1} = \frac{2}{3}.$$

Therefore, the vectors $\frac{\sqrt{3}}{\sqrt{2}}x$ and $\frac{\sqrt{3}}{\sqrt{2}}|x|$ form an orthonormal basis. So, the best approximation of the constant function 1 within the mentioned subspace can be found as:

$$p = \langle 1, \frac{\sqrt{3}}{\sqrt{2}}x \rangle \frac{\sqrt{3}}{\sqrt{2}}x + \langle 1, \frac{\sqrt{3}}{\sqrt{2}}|x| \rangle \frac{\sqrt{3}}{\sqrt{2}}|x| = \frac{3}{2}\langle 1, x \rangle x + \frac{3}{2}\langle 1, |x| \rangle |x| = \frac{3}{2}|x|.$$

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a singular value decomposition for M.

(b) Find the best rank 2 approximation of M.

REQUIRED KNOWLEDGE: **singular value decomposition, lower rank approximations**

SOLUTION:

(2a): Note that

$$M^T M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

As such, we obtain

$$\sigma_1 = \sigma_2 = \sigma_3 = \sqrt{3}.$$

This mans that

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By using the formula

$$u_i = \frac{1}{\sigma_i} M v_i$$

for $i = 1, 2, 3$, we get

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

By solving $M^T u_4 = 0$, we obtain

$$u_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, a singular value decomposition can be given by

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2b): One such approximation can be found as

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Suppose that a matrix has the characteristic polynomial

$$p(\lambda) = \lambda(\lambda + 2)(\lambda^2 + 1).$$

Prove that this matrix is

(a) singular.

(b) diagonalizable.

(c) NOT symmetric.

(d) NOT skew-symmetric.

(e) NOT orthogonal.

Required Knowledge: **eigenvalues/vectors, diagonalizability, (skew-)symmetric matrices, orthogonal matrices.**

Solution:

(3a): From the characteristic polynomial we can deduce that the eigenvalues of the matrix are 0, -2, i and $-i$. Since 0 is an eigenvalue, the matrix is singular.

(3b): Since the matrix has distinct eigenvalues, it is diagonalizable.

(3c): A symmetric matrix has real eigenvalues. Since this matrix has as i and $-i$ among its eigenvalues, the matrix can not be symmetric.

(3d): Suppose that a matrix A is skew-symmetric and let λ be an eigenvalue of A with a corresponding eigenvector x. In general, we have

$$x^* A x = x^* \lambda x = \lambda \|x\|^2$$
$$x^* A^T x = (A\bar{x})^T x = \bar{\lambda}\|x\|^2.$$

Since a skew-symmetric matrix satisfies $A^T = -A$, it follows that

$$\bar{\lambda}\|x\|^2 = x^* A^T x = -x^* A x = -\lambda \|x\|^2.$$

Hence, $\bar{\lambda} = -\lambda$, so $\mathrm{Re}(\lambda) = 0$. It follows that all eigenvalues of a skew-symmetric matrix are purely imaginary. Since the matrix in question has -2 among its eigenvalues, it can not be skew-symmetric.

Alternatively, if λ is an eigenvalue of A it is also an eigenvalue of A^T. If the matrix A is skew-symmetric, we have $A^T = -A$, and hence λ is also an eigenvalue of $-A$. Consequently, $-\lambda$ is an eigenvalue of A. So we see that if λ is an eigenvalue of a skew-symmetric matrix, then $-\lambda$ is as well. The matrix in question has -2 as one of its eigenvalues, but 2 is not one of its eigenvalues. Hence, the matrix is not skew-symmetric.

(3e): An orthogonal matrix is non-singular. Since this matrix is singular, it can not be orthogonal.

Alternatively, every eigenvalue λ of an orthogonal matrix satisfies $|\lambda| = 1$. The matrix in question has 0 and -2 as eigenvalues, which do not satisfy this condition, hence the matrix is not orthogonal.

Let a be a real number. Determine all values of a such that the matrix

$$\begin{bmatrix} 1 & a & 1 \\ a & a & a+1 \\ 1 & a+1 & 1 \end{bmatrix}$$

is

(a) positive definite.

(b) negative definite.

Required Knowledge: **Positive definite matrices, the principal minor test.**

Solution:

(4a): A symmetric matrix is positive definite if and only if all its principal minors are positive. Note that

$$\det(1) = 1 \qquad \det(\begin{bmatrix} 1 & a \\ a & a \end{bmatrix}) = a - a^2 \qquad \det(\begin{bmatrix} 1 & a & 1 \\ a & a & a+1 \\ 1 & a+1 & 1 \end{bmatrix}) = a + 2a(a+1) - a - a^2 - (a+1)^2 = -1.$$

Therefore, this matrix cannot be positive definite for any values of a

(4b): A symmetric matrix M is negative definite definite if and only if $-M$ is positive definite. Hence, we can apply the minor test for the negative of the matrix. Note that

$$\det(-1) = -1.$$

As such, this matrix is never negative definite.

Consider the matrix

$$\begin{bmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & -1 \\ 0 & -4 & 0 & 4 \end{bmatrix}.$$

(a) Show that the characteristic polynomial is $p(\lambda) = (\lambda - 2)^4$.

(b) Is it diagonalizable? Why?

(c) Put it into the Jordan canonical form.

REQUIRED KNOWLEDGE: **diagonalization, Jordan canonical form.**

SOLUTION:

(5a): Note that

$$\det\left(\begin{bmatrix} 2-\lambda & 2 & 0 & -1 \\ 0 & -\lambda & 0 & 1 \\ 1 & 5 & 2-\lambda & -1 \\ 0 & -4 & 0 & 4-\lambda \end{bmatrix}\right) = (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 2 & -1 \\ 0 & -\lambda & 1 \\ 0 & -4 & 4-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)^2 \det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)^2(-4\lambda + \lambda^2 + 4) = (2-\lambda)^4.$$

(5b): To compute eigenvalues, we need to solve the linear equations

$$\begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 1 & 5 & 0 & -1 \\ 0 & -4 & 0 & 2 \end{bmatrix} x = 0.$$

This is equivalent to

$$\begin{bmatrix} 1 & 5 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0.$$

Thus, there are two linearly independent eigenvectors, for instance

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Consequently, diagonalization is impossible.

(5c): Note that

$$\begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 1 & 5 & 0 & -1 \\ 0 & -4 & 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 1 & 5 & 0 & -1 \\ 0 & -4 & 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we try to solve one of the following line equations

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Clearly the latter has no solution whereas

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

solves the former. Thus, we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 1 & 5 & 0 & -1 \\ 0 & -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 1 \\ 1 & 5 & 0 & -1 \\ 0 & -4 & 0 & 2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

as a Jordan chain. Thus, we get

$$\begin{bmatrix} 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & -1 \\ 0 & -4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} & 0 & -3 \\ 0 & \frac{1}{2} & 0 & 1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & -3 \\ 0 & \frac{1}{2} & 0 & 1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$
